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APPROXIMATIONS AND BOUNDS FOR EIGENVALUES OF ELLIPTIC OPERATORS*

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1. Introduction. Let G be a bounded, n-dimensional domain with boundary Γ. Let $p(x)$ and $q_i(x)$, $i = 1, \cdots, n$, be positive functions defined on $\bar{G}$ and let D denote the linear, self-adjoint differential operator defined by

$$Du = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(q_i \frac{\partial u}{\partial x_i}\right).$$

The eigenvalue problem for D on G involves the nonzero solutions λ and $u(x)$ of

$$Du(x) + \lambda p(x)u(x) = 0, \qquad x \in G, \tag{1.1}$$

with the boundary condition

$$u(x) = 0, \qquad x \in \Gamma. \tag{1.2}$$

The eigenfunctions may be normalized so that

$$\frac{1}{M}\int_G u^2(x)p(x)\,dx = 1, \tag{1.3}$$

where

$$M = \int_G p(x)\,dx. \tag{1.4}$$

We are interested in computing accurate approximations to the eigenvalues and their corresponding eigenfunctions. Furthermore, we want to estimate the accuracy of our approximations by using them to compute upper and lower bounds for the true eigenvalues.

In §2 we prove the following theorem which is the basis for these upper and lower bound calculations.

THEOREM. *Let* λ^* *and* u^* *be an approximate eigenvalue and eigenfunction which satisfy* (1.1) *and* (1.3), *but not necessarily* (1.2). *Let*

$$\epsilon = \max_{x \in \Gamma} | u^*(x) | \tag{1.5}$$

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and assume $\epsilon < 1$. *Then there exists an eigenvalue* λ *of* D *on* G *satisfying*

$$\frac{|\lambda - \lambda^*|}{\lambda^*} \leqq \frac{\sqrt{2}\,\epsilon + \epsilon^2}{1 - \epsilon^2}. \tag{1.6}$$

The proof will use the fact that a maximum principle holds for the operator D; that is, if w is any function for which

$$Dw(x) = 0, \qquad x \in G, \tag{1.7}$$

then

$$\max_{x \in G} |w(x)| \leqq \max_{x \in \Gamma} |w(x)|. \tag{1.8}$$

In §3 we outline a general technique for constructing a λ^* and u^* which can produce an arbitrarily small ϵ in (1.5).

In §4 we specialize to two-dimensional domains and take D to be the Laplacian $\triangle$. We describe the method of collocation by interpolation for finding λ^* and u^*. This method is particularly suitable for domains with symmetry and domains with corners, that is, where a portion of Γ consists of two intersecting straight line segments.

In §5 we illustrate the method by applying it to elliptical domains of varying eccentricity.

In §6 we consider a much-studied example of a domain with corners, the L-shaped union of three unit squares. Its largest angle is $\frac{3}{2}\pi$ and thus is reentrant. For this example our method has proved significantly more accurate than any other known method. In fact, the upper and lower bounds obtained are even more precise than the approximations without bound obtained by other methods. For the first (smallest) eigenvalue of the L, for example, we show that

$$9.6397238 \leqq \lambda_1 \leqq 9.6397239.$$

Finite difference methods have been used and analyzed for these problems by Forsythe and Wasow [4], Fox [5], Moler [9], Veidinger [13] and others. However, convergence of these methods is quite slow for domains with reentrant corners. Difference methods in combination with collocation and conformal mapping have been used by Reid and Walsh [10] for the L, but their techniques become quite involved for more complicated domains.

The method of intermediate problems (A. Weinstein [14]) gives upper and lower bounds for certain domains. For example, Stadter [11] uses the method for rhombical domains.

Collocation methods similar to the one we describe in §4, but without the upper and lower bounds, have also been developed by J. Reid and B.

Martin of the University of Sussex and J. Walsh of the University of Manchester.

2. Proof of the theorem. The proof of our theorem is based on the following result which occurs in many forms in the literature. Collatz [2] attributes it to Kryloff and Bogoliubov and to D. H. Weinstein.

LEMMA. *Let A be a self-adjoint operator on a Hilbert space with inner product $(\cdot, \cdot)$. Let λ_n and u_n be the eigenvalues and orthonormal eigenfunctions of A. Assume $\{\lambda_n\}$ has no finite accumulation point. Let v be any element in the space spanned by $\{u_n\}$ and let*

$$\rho = \frac{(v, Av)}{\| v \|^2} \quad \textit{(the Rayleigh quotient)} \tag{2.1}$$

and

$$\sigma = \frac{\| Av \|}{\| v \|}. \tag{2.2}$$

Then $\sigma \geqq \rho$ and there exists at least one eigenvalue λ_k satisfying

$$\rho - \sqrt{\sigma^2 - \rho^2} \leqq \lambda_k \leqq \rho + \sqrt{\sigma^2 - \rho^2}. \tag{2.3}$$

For completeness we give the following proof. Let $v = \sum a_n u_n$. Then

$$\begin{aligned} \sigma^2 - \rho^2 &= \frac{\| (A - \rho I)v \|^2}{\| v \|^2} \\ &= \frac{\sum a_n^2 (\lambda_n - \rho)^2}{\sum a_n^2} \\ &\geqq \min_n (\lambda_n - \rho)^2 \\ &= (\lambda_k - \rho)^2 \quad \text{for some } k. \end{aligned}$$

Thus

$$| \lambda_k - \rho | \leqq \sqrt{\sigma^2 - \rho^2},$$

from which the conclusion (2.3) follows.

In proving the theorem, we take the operator A to be $-D$ and the inner product to be

$$(f, g) = \frac{1}{M} \int_a f(x) g(x) p(x)\, dx. \tag{2.4}$$

In the norm thus introduced the normalization becomes

$$\| u^* \| = 1.$$

The approximate eigenfunction u^* is not quite acceptable as the test

function v of the lemma because it is not zero on all of the boundary and hence is not in the space spanned by $\{u_n\}$. Let w be the solution to the boundary value problem

$$\begin{aligned} Dw(P) &= 0, && P \in G, \\ w(P) &= u^*(P), && P \in \Gamma, \end{aligned} \tag{2.5}$$

and let

$$v = u^* - w. \tag{2.6}$$

Then v satisfies the conditions of the lemma. We shall not need to actually calculate w, but shall only need to estimate its norm.

Defining

$$\omega = \| w \| \tag{2.7}$$

and

$$\cos\theta = \frac{(u^*, w)}{\| w \|}, \tag{2.8}$$

we find, after a short manipulation, that

$$\rho \pm \sqrt{\sigma^2 - \rho^2} = \lambda^* \frac{1 - \omega\cos\theta \pm \omega\sin\theta}{1 - 2\omega\cos\theta + \omega^2}. \tag{2.9}$$

Hence there exists $\lambda_k = \lambda$ with

$$\frac{|\lambda - \lambda^*|}{\lambda^*} \leqq \max \frac{|\omega\cos\theta \pm \omega\sin\theta - \omega^2|}{1 - 2\omega\cos\theta + \omega^2}, \tag{2.10}$$

where the maximum is over the two terms obtained with the $\pm$.

From the identity

$$\frac{\sqrt{2} + \omega}{1 - \omega^2} - \frac{\cos\theta + \sin\theta - \omega}{1 - 2\omega\cos\theta + \omega^2} = \frac{[\sqrt{2} - \cos\theta - \sin\theta - (\sqrt{2}\cos\theta - 1)\omega]^2}{(\sqrt{2} - \cos\theta - \sin\theta)(1 - \omega^2)(1 - 2\omega\cos\theta + \omega^2)}$$

and the fact that if $\omega^2 < 1$ the quantity on the right is nonnegative for all θ, we conclude

$$\frac{|\omega\cos\theta \pm \omega\sin\theta - \omega^2|}{1 - 2\omega\cos\theta + \omega^2} \leqq \frac{\sqrt{2}\omega + \omega^2}{1 - \omega^2} \quad \text{for all } \theta. \tag{2.11}$$

By (2.4), (1.8) and (2.5) we have

$$\omega = \| w \| \leqq \max_G |w| \leqq \max_\Gamma |u^*| = \epsilon. \tag{2.12}$$

Finally, combining (2.10)–(2.12) we find

$$\frac{|\lambda - \lambda^*|}{\lambda^*} \leqq \frac{\sqrt{2}\epsilon + \epsilon^2}{1 - \epsilon^2},$$

which establishes the theorem.

3. The method of particular solutions. In order to make use of the Theorem, we must have a constructive method for obtaining λ^* and u^*. One available approach can be based on the method of particular solutions developed independently by S. Bergman [1] and I. N. Vekua [12] (see also [6] and [7]). We explain it briefly for two-dimensional domains in which $q_1(x, y) = q_2(x, y)$. In this case (1.1) becomes

$$Lu \equiv \Delta u + (q_x/q)u_x + (q_y/q)u_y + \lambda(p/q)u = 0. \tag{3.1}$$

It is also assumed that q_x/q, q_y/q and p/q are entire analytic functions and that G is simply connected and contains the origin.

By writing the elliptic operator as a formally hyperbolic operator in the variables $z = x + iy$ and $z^* = x - iy$, and applying Riemann's method of integration, it is possible to set up in a natural way a linear one-to-one mapping $u = \Omega[f, f^*]$ of the pairs (f, f^*) of functions of the single complex variable z holomorphic in G and G^*, respectively, and satisfying $f(0) = f^*(0)$, onto the complex-valued functions u satisfying (3.1). The inverse mapping is defined by

$$f(z) = u\left(\frac{z}{2}, \frac{z}{2i}\right), \qquad f^*(z) = u\left(\frac{z}{2}, -\frac{z}{2i}\right).$$

By taking f and f^* to be powers of z, an infinite sequence of particular solutions is constructed:

$$\begin{aligned} &u_0(x, y) = \Omega[1, 1], \\ &u_{2n-1}(x, y) = \Omega[z^n, 0], \qquad u_{2n}(x, y) = \Omega[0, z^n], \quad n = 1, 2, \cdots. \end{aligned} \tag{3.2}$$

Finite linear combinations of these solutions

$$\sum_{j=0}^{N} c_j u_j(x, y) \tag{3.3}$$

may be used for the approximate eigenfunction. The operator L and hence the linear combination (3.3) depend upon λ. Thus λ and the c_j are chosen to make the ϵ in (1.5) as small as possible.

The mapping Ω is continuous in the Chebyshev norm in each compact subdomain of G. It thus follows from standard approximation theorems of one complex variable that any eigenfunction u regular in G can be approximated, uniformly in any compact subdomain of G, by the linear com-

binations (3.3). If G is a Jordan domain, and if u is sufficiently smooth on the closure of G, then it follows from a theorem of Walsh [12] that u can be approximated in the Chebyshev norm by linear combinations (3.3) even on the closure of G.

4. Collocation by interpolation. The remainder of the paper is concerned with two-dimensional domains and

$$D = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For this operator, the method of particular solutions produces

$$u^*(r, \theta) = \sum c_j J_{\alpha_j}(\sqrt{\lambda^*} r) e^{i\alpha_j \theta}, \tag{4.1}$$

where (r, θ) are polar coordinates in G and $J_\nu(\cdot)$ is the νth order Bessel function. The values of α_j, λ^* and c_j are to be determined.

There are two classes of domains for which the α_j are determined in a natural way. The first class includes domains which, when centered at the origin, are symmetric with respect to both the x and y axes. If $\alpha_j = 2j$, $j = 0, \cdots, N-1$, then

$$u^*(r, \theta) = \sum_{j=0}^{N-1} c_j J_{2j}(\sqrt{\lambda^*} r) \cos 2j\theta \tag{4.2}$$

retains the known symmetries of the first eigenfunction. Similar choices apply to other symmetries and higher eigenvalues.

The second class of domains involve those for which Γ contains a corner with angle $\frac{\pi}{\alpha}$ for some $\alpha \geqq \frac{1}{2}$. The origin is taken as the vertex of the angle, $\theta = 0$ and $\theta = \frac{\pi}{\alpha}$ correspond to the two straight line segments forming the angle and $\alpha_j = j\alpha, j = 1, \cdots, N$. Then

$$u^*(r, \theta) = \sum_{j=1}^{N} c_j J_{j\alpha}(\sqrt{\lambda^*} r) \sin j\alpha\theta \tag{4.3}$$

automatically satisfies the boundary condition on the sides of the angle and has the correct asymptotic singularity at the corner.

There are several possible methods for determining suitable values for λ^* and c_j. Possibly the simplest is to pick N points, $(r_k, \theta_k), k = 1, \cdots, N$, on the boundary Γ and interpolate the boundary condition at these points. Thus we require

$$u^*(r_k, \theta_k) = 0, \qquad k = 1, \cdots, N. \tag{4.4}$$

Using the u^* of (4.3) for illustration, if we let

$$\begin{aligned} a_{kj} &= a_{kj}(\lambda) = J_{\alpha_j}(\sqrt{\lambda} r_k) \sin \alpha_j \theta_k, \\ A(\lambda) &\text{ be the } N \times N \text{ matrix } \{a_{kj}(\lambda)\}, \end{aligned} \tag{4.5}$$

and

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix},$$

then (4.4) becomes

$$A(\lambda)c = 0. \tag{4.6}$$

We therefore define λ^* to be a root of the equation

$$\det\,(A(\lambda)) = 0. \tag{4.7}$$

For any root the approximate eigenfunction u^* is obtained by solving (4.6) for the coefficients c_j.

The resulting λ^* and u^* depend upon N and upon the choice of the interpolating points (r_k, θ_k). In our experiments we have usually chosen the points to be equally spaced along Γ. We have also found it helpful to impose conditions on the derivatives of u^* at certain points. These conditions are used to help force the contour $u^* = 0$ to be close to Γ and thereby reduce $\max_\Gamma | u^* |$.

The u^* defined in this way does not necessarily satisfy the normalization condition (1.3). A lower bound for $\| u^* \|$ is required. Let G_0 be the largest circle or circular sector centered at the origin and contained in G. Let

$$\delta = \text{radius of } G_0. \tag{4.8}$$

Then, again using (4.3) for illustration,

$$\| u^* \|^2 \geqq \frac{1}{M} \int_{G_0} (u^*(x))^2\, dx$$

$$= \frac{1}{M} \sum_{k,j=1}^{N} c_k c_j \int_0^\delta J_{\alpha_k}(\sqrt{\lambda^*} r) J_{\alpha_j}(\sqrt{\lambda^*} r) r\, dr \int_0^{\pi/\alpha} \sin \alpha_k \theta \sin \alpha_j \theta\, d\theta.$$

(Now, M is simply the area of G.) Since α_j is an integer times α, the set of functions $\sin \alpha_j \theta$, $j = 1, \cdots, N$, is orthogonal on $0 \leqq \theta \leqq \frac{\pi}{\alpha}$. Hence

$$\| u^* \|^2 \geqq \frac{1}{M} \cdot \frac{\pi}{2\alpha} \sum_{j=1}^{N} c_j^2 \int_0^\delta J_{\alpha_j}(\sqrt{\lambda^*} r)^2 r\, dr. \tag{4.9}$$

If u^* is given by (4.2), then α should be taken as $\frac{1}{2}$.

The Bessel function integrals occurring here can be expressed in terms of values at $r = \delta$ by using formulas proved, for example, in Courant-Hilbert [3]:

$$\int_0^\delta J_\nu(\mu r)^2 r\, dr = \frac{1}{2}\delta^2 \left\{ J_\nu'(\mu\delta)^2 + \left(1 - \frac{\nu^2}{\mu^2\delta^2}\right) J_\nu(\mu\delta)^2 \right\} \tag{4.10}$$

and

$$J_\nu'(\mu\delta) = \frac{\nu}{\mu\delta} J_\nu(\mu\delta) - J_{\nu+1}(\mu\delta). \tag{4.11}$$

The important quantity occurring in the upper and lower bounds is, of course, $\epsilon = \max_{x\in\Gamma} | u^*(x) |$. For a given domain it may be possible to obtain a good estimate of this a priori (once the distribution of the N boundary points is specified). However, we have chosen to calculate the maximum a posteriori. But this involves an additional theoretical problem: What is the error made in the computation of the error bounds? A complete rigorous analysis is, in principle, possible and would include a detailed study of the algorithm used to find $\max | u^* |$ as well as the roundoff errors in the other computations.

5. Elliptical domains. Application to domains with two-fold symmetry can be illustrated with ellipses of various eccentricities. As may be expected, the method is very successful. The bounds obtained for the first eigenvalue are given in Table 1. For all values of the major axis, $N = 10$ was used in (4.2). The 10 interpolating points were equally spaced along one-fourth of the perimeter. Even tighter bounds would be obtained with more points or with a more appropriate distribution of the 10 points.

The eigenvalues of ellipses can also be characterized as the zeroes of certain Mathieu functions. We thus have here a practical method for computing these zeroes.

6. The L-shaped membrane. As an important example, we consider the

TABLE 1

Bounds for the first eigenvalue of ellipses

(*Semiminor axis* = 1.0)

Semimajor axis	Lower bound Upper bound
1.0	5.7831859^{6}_{7}
1.1	5.2800383^{3}_{4}
1.2	4.8952213^{3}_{5}
1.4	4.3534183^{6}_{9}
1.6	3.9968064^{7}_{9}
1.8	3.7481592^{2}_{3}
2.0	3.566726^{58}_{62}
3.0	3.108128^{66}_{75}
4.0	2.92025^{080}_{120}
5.0	2.8180^{6952}_{7142}
10.0	2.63^{238456}_{307909}

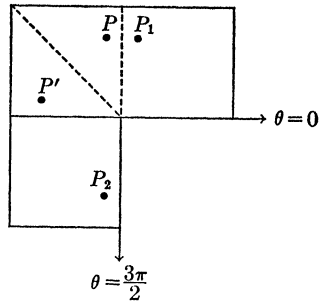

FIG. 1. *Symmetries in the L-shaped membrane*

L-shaped membrane illustrated in Fig. 1. With the polar coordinates centered at the reentrant corner, θ runs from 0 to $\frac{3\pi}{2}$. Hence

$$\alpha = \tfrac{2}{3}. \tag{6.1}$$

The above methods could be applied directly, but the amount of computation required is decreased if symmetries are used. First, there is the obvious symmetry about the center line. Let $P = (r, \theta)$ be any point in G and $P' = \left(r, \frac{3\pi}{2} - \theta\right)$ be its reflection about the center line $\theta = \frac{3\pi}{4}$. Then any eigenfunction must satisfy (or, in the case of a multiple eigenvalue, can be chosen to satisfy) either

$$u(P) = u(P') \tag{6.2}$$

or

$$u(P) = -u(P'). \tag{6.3}$$

J. Hersch, in [8], establishes the existence of additional symmetry about the lines which divide the L into its three component unit squares. Let

$$P = (r, \theta), \qquad \frac{\pi}{2} \leqq \theta \leqq \pi,$$

be a point in the "middle" square and let

$$P_1 = (r, \pi - \theta) \quad \text{and} \quad P_2 = (r, 2\pi - \theta)$$

be the reflections of P about the lines $\theta = \frac{\pi}{2}$ and $\theta = \pi$. (See Fig. 1.) Then Hersch shows that any eigenfunction must satisfy either

$$u(P) = u(P_1) + u(P_2) \tag{6.4}$$

or

(6.5) u is an eigenfunction of the unit square.

These symmetries have two consequences. First, we need only work in the "first" square, $0 \leqq \theta \leqq \frac{\pi}{2}$. Any approximation of u there can automatically be extended to the other two squares. Second, requiring the approximate eigenfunction u^* to have the same symmetries as u substantially reduces the number of terms necessary to obtain a given accuracy. For example, if u^* is made to satisfy (6.2) and (6.4), then it follows that

$$\sum_{j=1}^{N} c_j J_{\alpha_j}(\sqrt{\lambda_N} r) \sin \alpha_j \theta \left(2 \cos \alpha_j \frac{\pi}{2} - 1\right) = 0 \tag{6.6}$$

for all r and θ. Hence, if $c_j \neq 0$,

$$\cos \alpha_j \frac{\pi}{2} = \frac{1}{2}$$

and

$$\alpha_j = \alpha, 5\alpha, 7\alpha, 11\alpha, 13\alpha, \cdots. \tag{6.7}$$

Thus, for a given N, replacing the original definition of α_j (namely, $\alpha_j = j\alpha$) with the above sequence increases the number of terms with nonzero c_j by a factor of three. Furthermore, u^* is guaranteed to exhibit the proper symmetries.

The sequence (6.7) follows from (6.2) and (6.4). These equations hold, for example, for the eigenfunction corresponding to the first eigenvalue. If, instead, we want u^* to satisfy (6.3) and (6.4), we find

$$\alpha_j = 2\alpha, 4\alpha, 8\alpha, 10\alpha, 14\alpha, \cdots \tag{6.8}$$

or, if u^* satisfies (6.5),

$$\alpha_j = 3\alpha, 6\alpha, 9\alpha, \cdots. \tag{6.9}$$

The boundary points (r_k, θ_k) chosen on the perimeter of the first square are spaced at an interval h, where $1/h$ is an integer. This defines $2/h$ points:

$$\begin{aligned} (r_1, \theta_1) &= (\sqrt{1 + h^2}, \tan^{-1} h), \\ (r_2, \theta_2) &= (\sqrt{1 + 4h^2}, \tan^{-1} 2h), \\ &\cdots \\ (r_{2/h}, \theta_{2/h}) &= (1, \pi/2). \end{aligned}$$

In addition we require

$$\frac{\partial u^*}{\partial \theta} = 0$$

at the two corners $(1, 0)$ and $(\sqrt{2}, \pi/4)$. Thus the total number of terms is

$$N = \frac{2}{h} + 2.$$

Table 2 lists λ^* and ϵ for the first eigenvalue of the L obtained with various values of h. It will be noted that ϵ, and hence the gap between the upper and lower bounds, actually increases with the larger values of N. This is apparently due to roundoff errors in the solution of (4.6) for the coefficients c_j. These errors do not mean that the computed bounds are not, in fact, bounds, only that they are farther apart than they need be.

Table 3 lists the best bounds obtained for ten eigenvalues of the L. (A separate calculation based on [16] proves they are the *first* ten.) The mode of the eigenfunction is classified as symmetric or antisymmetric and as "square" for functions which satisfy (6.2), (6.3) and (6.5), respectively. The exact values for the "square" eigenfunctions are $\lambda_3 = 2\pi^2$ and the double eigenvalue $\lambda_8 = \lambda_9 = 5\pi^2$. The calculated values of λ^* agree with these exact values to the number of decimal places used in the table.

By way of comparison with other methods, the best values obtained by Reid and Walsh [10] are $\lambda_1 = 9.63972$ and $\lambda_2 = 15.1973$.

C. Moler, together with Professor G. E. Forsythe of Stanford University, obtained a value (unpublished) of 9.639724 for λ_1 by extrapolation of finite difference values obtained with very small meshes. Neither of these other methods yields any bound of its accuracy.

TABLE 2

Dependence of λ^* and ϵ upon N

h	N	λ^*	ϵ
1	4	9.658161723	$1.02_{10^{-2}}$
$\frac{1}{2}$	6	9.639624491	$1.56_{10^{-4}}$
$\frac{1}{3}$	8	9.639726632	$1.58_{10^{-5}}$
$\frac{1}{4}$	10	3703	$1.57_{10^{-6}}$
$\frac{1}{5}$	12	3855	$1.74_{10^{-7}}$
$\frac{1}{6}$	14	3844	$3.92_{10^{-8}}$
$\frac{1}{7}$	16	3844	$2.62_{10^{-9}}$
$\frac{1}{8}$	18	3845	$7.11_{10^{-9}}$
$\frac{1}{9}$	20	3845	$2.95_{10^{-8}}$
$\frac{1}{10}$	22	3845	$3.74_{10^{-9}}$
$\frac{1}{11}$	24	3844	$4.52_{10^{-8}}$
$\frac{1}{12}$	26	3846	$1.49_{10^{-7}}$

TABLE 3

Bounds for the first ten eigenvalues of the L

n	Lower bound for λ_n Upper bound for λ_n	Mode of eigenfunction
1	9.6397238^{05}_{84}	symmetric
2	15.19725^{184}_{201}	antisymmetric
3	19.73920^{85}_{91}	symmetric, square
4	29.52148^{04}_{18}	antisymmetric
5	31.91263^{31}_{88}	symmetric
6	41.4745^{039}_{159}	symmetric
7	44.948^{467}_{509}	antisymmetric
8, 9	49.3480^{09}_{35}	multiple, square
10	56.7096^{02}_{18}	symmetric

7. Other domains. In all fairness, it should be reported that results are not always as satisfactory as these examples indicate. Experiments with rhombical domains were also made. These domains have both symmetry and a corner. However, we have been unable to obtain bounds significantly tighter than those obtained in [11] by other methods. The difficulty stems from the existence of *two* corners with nonintegral values of α. These singularities are reflected in the slow convergence of either (4.2) or (4.3) and consequent influence of roundoff errors in the coefficients. Other methods for defining and computing these coefficients are currently being investigated.

8. Computational details. We conclude with a few remarks on the techniques used in the computations.

The Bessel functions are evaluated by truncating the alternating series

$$J_\nu(x) = x^\nu \sum_{n=0}^{\infty} t_n \left(-\frac{x^2}{4}\right)^n, \tag{8.1}$$

where $t_n = t_{n-1}/(n(n+\nu))$. The value of t_0 should be

$$t_0 = t_0(\nu) = \frac{1}{2^\nu \Gamma(\nu+1)}, \tag{8.2}$$

but it is not necessary to compute this gamma function since t_0 is merely a scale factor which can be absorbed into the coefficients c_j. In fact, it is convenient to use a value of t_0 quite different from that given by (8.2) in order to scale J_ν to be about 1 for the arguments used. If this is done, it is not necessary to check for floating point exponent underflow and overflow during the evaluation of the determinant (4.7).

(With a different value of $t_0(\nu)$, (4.12) should be replaced by

$$J_\nu'(\mu\delta) = \frac{\nu}{\mu\delta} J_\nu(\mu\delta) - \frac{t_0(\nu)}{2(\nu+1)\cdot t_0(\nu+1)} J_{\nu+1}(\mu\delta).) \tag{4.11'}$$

The function det $(A(\lambda))$ is computed, using triangular decomposition (Gaussian elimination) with partial pivoting, for a few values of λ near a zero of the function. Then inverse interpolation is used several times to find λ^*.

This use of inverse interpolation proves to be quite satisfactory for the smaller eigenvalues of the L. But for the higher eigenvalues and for all the eigenvalues of some other domains studied, the quantity det $(A(\lambda))$ has a local extremum very near λ^* and it is therefore necessary to locate λ^* quite accurately before inverse interpolation can be used.

Inverse iteration [15] is used to solve (4.6) for the coefficients. In fact, the required triangular decomposition of $A(\lambda^*)$ has already been computed during the determinant evaluation. Any reduction of the roundoff errors incurred in the computed c_j at this point would lead to tighter upper and lower bounds.

Finally, the calculation of $\max | u^*(P) |$ on Γ is a one-dimensional maximizing problem. The boundary is broken into intervals by the points (r_k, θ_k). It is assumed that the only local minima of $| u^* |$ occur at these points and thus a simple maximizing search is used on each interval. If another local minimum is found, the search can be carried out on each of the resulting subintervals.

Our computer program is written in ALGOL. A Control Data 1604A with a 36-bit floating point significand was used. The calculation of one value of λ^* and the resulting bounds takes slightly over one minute for the largest values of N shown. Most of this time is spent in the maximizing searches on the boundary. Even so, this time represents a considerable reduction over the other methods which do not produce bounds.

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REFERENCES

[1] S. Bergman, *Functions satisfying certain partial differential equations of elliptic type and their representation*, Duke Math. J., 14 (1947), pp. 349-366.

[2] L. Collatz, *Eigenwertprobleme und ihre Numerische Behandlung*, Akademische Verlagsgesellschaft, Leipzig, 1945, p. 208.

[3] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 1, Interscience, New York, 1953.

[4] G. E. Forsythe and W. R. Wasow, *Finite-Difference Methods for Partial Differential Equations*, John Wiley, New York, 1962.

[5] L. Fox, *Numerical Solution of Ordinary and Partial Differential Equations*, Pergamon Press, Oxford, 1962.

[6] P. HENRICI, *Zur Funktionentheorie der Wellengleichung*, Comm. Math. Helv., 27 (1953), pp. 235–293.

[7] ———, *A survey of I. N. Vekua's theory of elliptic partial differential equations with analytic coefficients*, Z. Angew. Math. Physik, 8 (1957), pp. 169–203.

[8] J. HERSCH, *Erweiterte Symmetrieeigenschaften von Lösungen gewisser linearer Rand- und Eigenwertprobleme*, J. Reine Angew. Math., 218 (1965), pp. 143–158.

[9] C. MOLER, *Finite difference methods for the eigenvalues of Laplace's operator*, Report CS 22, Stanford University Computer Science Dept., 1965.

[10] J. K. REID AND J. E. WALSH, *An elliptic eigenvalue problem for a reentrant region*, J. Soc. Indust. Appl. Math., 13 (1965), pp. 837–850.

[11] J. T. STADTER, *Bounds to eigenvalues of rhombical membranes*, Ibid., 14 (1966), pp. 324–341.

[12] I. N. VEKUA, *Novya Metody Rešenija Ellipticeskikh Uravnenij*, (New Methods for Solving Elliptic Equations), O G I Z, Moscow and Leningrad, 1948.

[13] L. VEIDINGER, *Computation of the eigenvalues of a membrane by finite difference methods*, Zh. Vychisl. Mat. i Mat. Fiz., 4 (1964), pp. 1037–1044.

[14] A. WEINSTEIN, *Some numerical results in intermediate problems for eigenvalues*, Numerical Solution of Partial Differential Equations, J. Bramble, ed., Academic Press, New York, 1966.

[15] J. H. WILKINSON, *The Algebraic Eigenvalue Problem*, Oxford University Press, Oxford, 1965.

[16] J. HERSCH, *Lower bounds for all eigenvalues by cell functions: a refined form of H. F. Weinberger's method*, Arch. Rational Mech. Anal., 12 (1963), pp. 361–366.